

The Effect of Particles on Blast Waves in a Dusty Gas

Fumio Higashino *

Stoßwellenlabor, Institut für Luft- und Raumfahrt, Technische Hochschule Aachen

and Tateyuki Suzuki**

Department of Aeronautics, University of Tokyo

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The effect of suspended particles on cylindrical blast waves is investigated by making use of a series expansion method with respect to $1/M^2$ take off [2], where M is the shock Mach number. Velocity and temperature equilibrium between the gas and the particles is assumed. As a result, one finds that the particles are assembled near the shock front and that behind it the slopes of the pressure and velocity profiles increase, as the volume fraction of particles increases. The difference between the zeroth and first order solutions becomes the smaller, the larger the volume fraction of the particles. The decay rate of a blast wave in a dusty gas is greater than in a dust-free gas.

1. Introduction

Blast waves or detonation waves in particle-laden gases occur in such areas as coal-dust-gas explosions [1], metallized propellant rockets [2] and supersonic flight in polluted air [3]. In the propagation of a shock wave through a dust-laden-gas the dust particles take up momentum. This may affect the velocity of the shock and the flow field behind it, that is, the structure of the blast wave. In spite of many studies on blast waves in a dust-free gas, only a few analyses about heterogeneous systems [4], [5] exist. Gerber and Bartos [5] have treated the problem by neglecting both the interaction of particles and the effects of the particles on the gas flow. Then the problem reduces to the computation of trajectories of individual dust particles in a known unsteady flow field. In general the energy and momentum exchange between the gas and the particles should be taken into account, since the gas phase flow field is different from the corresponding one in a dust-free atmosphere. However, as will be seen later, the interaction terms vanish for a flow with temperature and velocity equilibrium.

We have already analysed the problem of a strong blast wave under the assumption of temperature and velocity equilibrium and got a self-similarity solution as the zeroth approximation. In the present work, the analysis is extended to the first order approximation under the same assumption regarding interaction terms between the gas and the particles as before [6]. From the order estimation of the inhomogeneous terms in the basic equations one can find that there is a characteristic dimensionless parameter which is the ratio of the shock position to the radius of particles. Usually, the size of the particles is in the order of $10\text{ }\mu\text{m}$ [7]. In this case, when the shock position is greater than 10 cm, the assumption of temperature and velocity equilibrium between the gas and the particles may be a good approximation. In addition, the basic equations are much simplified.

As a typical example, line source explosions in a dusty gas are investigated under the assumptions of velocity and temperature equilibrium. The method is based on the series expansion method of Sakurai [8]. From the zeroth order solution one can get a self-similarity solution for cylindrical blast waves in a dusty gas. As a result one finds that the dust is assembled near the shock front and that this tendency becomes more pronounced as the volume fraction of the particles increases. The first order solution reveals that the decay rate of a blast wave in a dusty gas is greater than in a dust-free gas. The profiles of the pressure, density, and flow velocity for finite shock strength approach those for the self-similarity solution.

* On leave from Department of Mechanical Engineering, Tokyo-Noko University, Koganei-shi, Tokyo, Japan.

** Saitama Institute of Technology.

*** First version was received October 6, 1976.

Reprint requests to Prof. F. Higashino, Department of Mechanical Engineering, Tokyo-Noko University, Koganei-shi, Tokyo, 184/Japan.

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2. Basic Equations and Assumptions

Because of the intricacy and lack of information about the phenomena, one is compelled to make assumptions for solving the problem. The following assumptions are made for the dust particles and their interaction with the gas flow (as in the previous study [6]):

- (1) The mixture of gas and particles is treated as a continuum.
- (2) The diameter of a particle is negligibly small in comparison with the characteristic hydrodynamic length, e.g., the distance between the shock position and the center of explosion.
- (3) The number density of the particles is small in comparison with that of the gas molecules.
- (4) The Brownian motion of the particles may be negligibly small [7].
- (5) The particles do not interact with one another.
- (6) The gas follows the perfect gas law and is non-viscous and adiabatic. Even if boundary layer effects are neglected, however, the viscosity of the gas enters into the drag force between the gas and the particles, and also heat transfer exists between them.
- (7) The temperature distribution within a particle is taken to be uniform, and the specific heats both of the gas and the particles are constant.
- (8) Deformations and phase changes, such as evaporation and condensation of the particles, are neglected.
- (9) Temperature and velocity equilibrium between the gas and the particles is assumed.

For a two phase medium with the above assumptions the basic equations may be expressed as follows:

continuity of the gas

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial \rho_g u_g}{\partial \chi} + j \frac{\rho_g u_g}{\chi} = 0, \quad (1)$$

continuity of the particles

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial \rho_p u_p}{\partial \chi} + j \frac{\rho_p u_p}{\chi} = 0, \quad (2)$$

conservation of momentum for the mixture

$$\begin{aligned} \rho_g \frac{\partial u_g}{\partial t} + \rho_p \frac{\partial u_p}{\partial t} + \rho_g u_g \frac{\partial u_g}{\partial \chi} \\ + \rho_p u_p \frac{\partial u_p}{\partial \chi} + \frac{\partial p}{\partial \chi} = 0, \end{aligned} \quad (3)$$

conservation of momentum for the particles

$$\rho_p \frac{\partial u_p}{\partial t} + \rho_p u_p \frac{\partial u_p}{\partial \chi} + \varepsilon \frac{\partial p}{\partial \chi} = F, \quad (4)$$

conservation of energy for the mixture

$$\begin{aligned} \frac{\partial}{\partial t} (\rho_g e_g + \rho_p e_p) + \frac{\partial}{\partial \chi} \\ \cdot \{ \rho_g u_g e_g + (1 - \varepsilon) p u_g + \rho_p u_p e_p + \varepsilon p u_p \} + \frac{j}{\chi} \\ \cdot \{ \rho_g u_g e_g + (1 - \varepsilon) p u_g + \rho_p u_p e_p + \varepsilon p u_p \} = 0, \end{aligned} \quad (5)$$

conservation of energy for the particles:

$$c \frac{\partial T_p}{\partial t} + c u_p \frac{\partial T_p}{\partial \chi} = Q, \quad (6)$$

with a set of constitutive equations

$$p = \bar{\rho}_g R T, \quad (7)$$

$$e_g = \frac{1}{2} u_g^2 + c_v T_g, \quad (8)$$

$$e_p = \frac{1}{2} u_p^2 + c T_p, \quad (9)$$

$$\rho_g = (1 - \varepsilon) \bar{\rho}_g, \quad (10)$$

$$\rho_p = \varepsilon \bar{\rho}_p, \quad (11)$$

$$\varepsilon = \frac{4}{3} \pi a_p^3 n_p, \quad (12)$$

$$F = \frac{\pi}{2} a_p^2 C_D \bar{\rho}_g (u_g - u_p) |u_g - u_p| n_p, \quad (13)$$

$$Q = \frac{6 h_l}{2 a_p \bar{\rho}_p} (T_g - T_p). \quad (14)$$

The symbols used here are the usual ones and you can find the detailed meaning of the symbols in [4]. For the flow with velocity and temperature equilibrium the following equations are used

$$u_g = u_p = u, \quad (15)$$

$$T_g = T_p = T, \quad (16)$$

in place of (4) and (6). After some calculations and by using (15) and (16), the basic equations are reduced to the following ones:

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial \rho_g u}{\partial \chi} + j \frac{\rho_g u}{\chi} = 0, \quad (17)$$

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial \rho_p u}{\partial \chi} + j \frac{\rho_p u}{\chi} = 0, \quad (18)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial \chi} + \frac{1}{\rho_p + \rho_g} \frac{\partial p}{\partial \chi} = 0, \quad (19)$$

$$\frac{\partial}{\partial t} \left[(\rho_g + \rho_p) \frac{u^2}{2} + (c_v \rho_g + c \rho_p) T \right] + \left(\frac{\partial}{\partial \chi} + \frac{j}{\chi} \right) \cdot \left[(\rho_g + \rho_p) \frac{u^2}{2} + (c_v \rho_g + c \rho_p) T + p u \right] = 0, \quad (20)$$

with

$$(1 - \varepsilon) p = \rho_g R T. \quad (21)$$

The boundary conditions across the shock discontinuity are given by the generalized Rankine-Hugoniot relation including the effect of the finite volume fraction of the particles. They may be expressed as

$$m_g = (\rho_g)_0 \dot{R}_s = \rho_g (\dot{R}_s - u), \quad (22)$$

$$m_p = (\rho_p)_0 \dot{R}_s = \rho_p (\dot{R}_s - u), \quad (23)$$

$$(m_g + m_p) \dot{R}_s + p_0 = (m_g + m_p) (\dot{R}_s - u) + p, \quad (24)$$

$$\begin{aligned} (m_g + m_p) \frac{\dot{R}_s^2}{2} + (c_v m_g + c m_p) T_0 + p_0 \dot{R}_s \\ = (m_g + m_p) \frac{(\dot{R}_s - u)^2}{2} \\ + (c_v m_g + c m_p) T + p (\dot{R}_s - u), \end{aligned} \quad (25)$$

where the subscript zero denotes physical quantities ahead of the shock and \dot{R}_s is the shock speed. From (22) and (23) one can get

$$\frac{\rho_g}{(\rho_g)_0} = \frac{\rho_p}{(\rho_p)_0}. \quad (26)$$

This relation should be satisfied by the basic equations. Substituting (26) into (17) and (18), one can easily find that (18) is identical with (17). Thus the differential equations to be solved are reduced to the equations

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial \rho_g u}{\partial \chi} + j \frac{\rho_g u}{\chi} = 0, \quad (27)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial \chi} + \frac{1}{\rho_g + \rho_p} \frac{\partial p}{\partial \chi} = 0, \quad (28)$$

$$\begin{aligned} (1 + \gamma \delta \eta) \left(\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial \chi} \right) - \frac{\gamma}{1 - \varepsilon} (1 + \delta \eta) \\ \cdot \frac{p}{\rho_g} \left(\frac{\partial \rho_g}{\partial t} + u \frac{\partial \rho_g}{\partial \chi} \right) = 0. \end{aligned} \quad (29)$$

Here δ and η are the specific heat ratio and the initial loading ratio, respectively, and defined as

$$\delta = \frac{c}{c_p}, \quad \eta = \frac{\rho_p}{\rho_g} = \frac{(\rho_p)_0}{(\rho_g)_0}. \quad (30)$$

3. Analysis

It is convenient to introduce dimensionless variables (ξ, M) as independent variables instead of (χ, t) defined by

$$\xi = \chi / R_s, \quad M = \dot{R}_s / a_0, \quad (31)$$

where a_0 is the sound velocity ahead of the shock and M the shock Mach number. The dependent variables of the flow are also transformed into the nondimensional pressure f , density g , and flow velocity h as

$$p(\chi, t) = (\rho_g)_0 \dot{R}_s^2 f(\xi, M), \quad (32)$$

$$\rho_g(\chi, t) = (\rho_g)_0 g(\xi, M), \quad (33)$$

$$u(\chi, t) = \dot{R}_s h(\xi, M). \quad (34)$$

With these dimensionless variables, the basic differential equations are expressed as

$$(h - \xi) \frac{\partial g}{\partial \xi} + g \frac{\partial h}{\partial \xi} + \theta g \left(\frac{M}{g} \frac{\partial g}{\partial M} + \frac{j}{\theta} \frac{h}{\xi} \right) = 0, \quad (35)$$

$$\begin{aligned} (h - \xi) \frac{\partial h}{\partial \xi} + \frac{1}{1 + \eta} \cdot \frac{1}{g} \cdot \frac{\partial f}{\partial \xi} \\ + \theta h \left(1 + \frac{M}{h} \frac{\partial h}{\partial M} \right) = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} (h - \xi) \frac{\partial f}{\partial \xi} + \gamma \frac{1 + \eta \delta}{1 + \gamma \eta \delta} \cdot \frac{1}{1 - \varepsilon g} \left(\frac{\partial h}{\partial \xi} + j \frac{h}{\xi} \right) \\ + \theta f \left(2 - \frac{M}{f} \frac{\partial f}{\partial M} \right) = 0, \end{aligned} \quad (37)$$

where θ is the decay coefficient of blast waves and defined by

$$\theta = \frac{R_s}{M} \cdot \frac{dM}{dR_s}. \quad (38)$$

Though there are several methods to solve the above partial differential equations, we adopted the series expansion method with respect to $1/M^2$ by Sakurai, since in this method it is easy to estimate the effect of counterpressure qualitatively. Then the independent variables are expanded into the forms

$$f(\xi, M) = f_0(\xi) + f_1(\xi) \frac{1}{M^2} + \cdots, \quad (39)$$

$$g(\xi, M) = g_0(\xi) + g_1(\xi) \frac{1}{M^2} + \cdots, \quad (40)$$

$$h(\xi, M) = h_0(\xi) + h_1(\xi) \frac{1}{M^2} + \cdots, \quad (41)$$

and also for the decay coefficient

$$\theta(M) = \theta_0 + \theta_1 \frac{1}{M^2} + \dots \quad (42)$$

Substituting (39) to (42) into (35) to (37) and equating the coefficients of terms with corresponding powers of $1/M^2$, one can get the following systems of ordinary differential equations.

For the zeroth order approximation one can get

$$(h_0 - \xi) \frac{dg_0}{d\xi} + g_0 \frac{dh_0}{d\xi} + j \frac{g_0 h_0}{\xi} = 0, \quad (43)$$

$$(h_0 - \xi) \frac{dh_0}{d\xi} + \frac{1}{(1 + \eta)g_0} \frac{df_0}{d\xi} + \theta_0 h_0 = 0, \quad (44)$$

$$(h_0 - \xi) \frac{df_0}{d\xi} + \frac{\Gamma f_0}{(1 - \varepsilon g_0)} \cdot \left(\frac{dh_0}{d\xi} + j \frac{h_0}{\xi} \right) + 2\theta_0 h_0 = 0 \quad (45)$$

where Γ is given by

$$\Gamma = \frac{\gamma(1 + \eta\delta)}{(1 - \varepsilon g_0)(1 + \gamma\eta\delta)}. \quad (46)$$

From (22) to (25) the corresponding boundary conditions are expressed as

$$h_0(1) = 1 - \frac{\gamma - 1}{\gamma + 1 + 2\gamma\eta\delta} \cdot \left[1 - 2\varepsilon + \frac{2\varepsilon\gamma(1 + \eta\delta)}{\gamma - 1} \right], \quad (47)$$

$$g_0(1) = \frac{1}{1 - h_0(1)}, \quad (48)$$

$$f_0(1) = (1 + \eta)h_0(1), \quad (49)$$

$$h_0(0) = 0. \quad (50)$$

In the self-similarity solutions the decay coefficient θ_0 is equal to $-(j+1)/2$, which is derived from the conservation of energy within a blast wave.

For the flow of finite shock strength one can get a solution from the first order approximation. The corresponding equations are given as

$$(h_0 - \xi) \frac{dg_1}{d\xi} + g_0 \frac{dh_1}{d\xi} + h_1 \frac{dg_0}{d\xi} + g_1 \frac{dh_0}{d\xi} + \frac{j}{\xi} \cdot (g_0 h_1 + g_1 h_0) - 2\theta_0 g_1 = 0, \quad (51)$$

$$(h_0 - \xi) g_0 \frac{dh_1}{d\xi} + \frac{1}{1 + \eta} \frac{df_1}{d\xi} - g_0 h_1 - g_1 (h_0 - \xi) + \theta_0 (g_1 h_0 - g_0 h_1) + \theta_1 g_0 h_0 = 0, \quad (52)$$

$$(h_0 - \xi) \frac{df_1}{d\xi} + \frac{\Gamma f_0}{(1 - \varepsilon g_0)} \frac{dh_1}{d\xi} + \left[h_1 - \frac{\varepsilon g_1 (h_0 - \xi)}{1 - \varepsilon g_0} \right] \frac{df_0}{d\xi} + \frac{\Gamma f_1}{1 - \varepsilon g_0} \frac{dh_0}{d\xi} + \frac{j\Gamma(h_1 f_0 + h_0 f_1)}{\xi(1 - \varepsilon g_0)} - \frac{2\varepsilon g_1 f_0 \theta_0}{1 - \varepsilon g_0} + 2f_0 \theta_1 = 0, \quad (53)$$

with the boundary conditions,

$$h_1(1) = -\frac{2(1 + \eta\delta)}{(\gamma + 1 + 2\gamma\eta\delta)(1 + \eta)(1 - \varepsilon)}, \quad (54)$$

$$g_1(1) = g_0^2 h_1(1), \quad (55)$$

$$f_1(1) = \frac{1}{\gamma(1 - \varepsilon)} + (1 + \eta)h_1(1), \quad (56)$$

$$h_1(0) = 0. \quad (57)$$

In the case of the zeroth order solution one can easily find that the decay coefficient θ_0 is equal to -1 for cylindrical flows. For the first order solution, however, the value of the decay coefficient remains as an unknown. To determine the correct value of θ_1 which may approximately seem to satisfy the condition of $h_1(0)=0$ at the center, the iteration method was used.

These two sets of ordinary differential equations with respect to ξ are easily integrated by a usual numerical method with the corresponding boundary conditions. The integration was carried out from $\xi=1$ at the shock front to $\xi=0$ at the center of explosions by the Runge-Kutta-Gill method.

4. Results and Discussion

Line source explosions in a dusty gas are investigated under the assumptions of velocity and temperature equilibrium between the gas and the particles.

As an example, numerical calculations have been made for the case of $\gamma=1.4$, $\delta=c/c_p=1.0$ and $\zeta=1000$. Here ζ is the specific material density ratio and related to the initial loading ratio η by $\zeta=(1-\varepsilon)\eta/\varepsilon$. This case may be realized in an air flow with a suspension of liquid droplets. The volume fraction of particles is varied from $\varepsilon=0$ to 0.004.

The density, velocity and pressure profiles for the case of zeroth order approximation, which provides the self-similarity solution, are shown in Figs. 1—3,

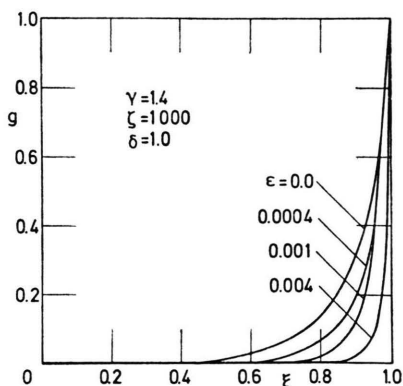


Fig. 1. The profiles of nondimensional density behind shock front for the zeroth order solution.

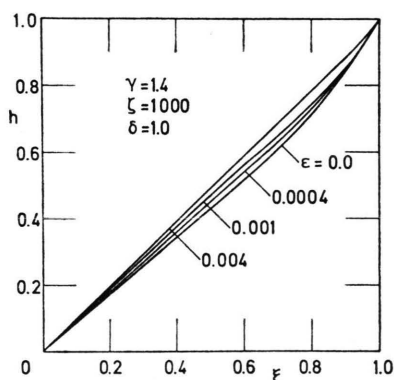


Fig. 2. The profiles of nondimensional velocity behind shock front for the zeroth order solution.

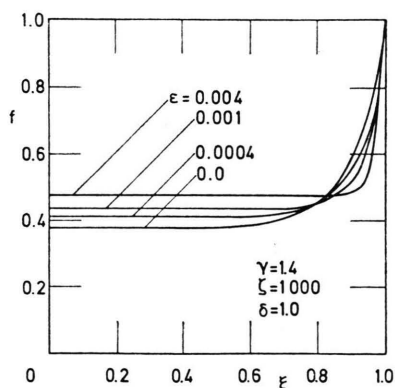


Fig. 3. The profiles of nondimensional pressure behind shock front for the zeroth order solution.

respectively. In these figures the physical quantities are normalized by the values at the shock front. From the figures one can find that the material is concentrated near behind the shock front as the volume fraction of the particles ϵ increases. This means the substantial thickness of a blast wave becomes thinner with increasing ϵ . This phenomenon agrees qualitatively with the result by Gerber and Bartos and may be explained by considering the inertia of the particles. This tendency is exaggerated under the assumed velocity and temperature equilibrium between the gas and the particles. Further discussions of the pressure, density and velocity profiles behind the shock for the self-similarity solution can be seen in the previous report in detail [6].

Figure 4 shows a comparison of the structures of blast waves between the finite and infinite shock strength. The broken lines show the case of infinite shock strength, which is given by the self-similarity law. The solid lines show the results for a finite shock strength of $M = 4$. The profiles of the non-dimensional pressure, density and particle velocity for both cases are very similar to each other. Especially, the velocity profile for $M = 4$ is overlapped by that for the self-similarity solution. We have also calculated the case of $M = 6$. However, the results are very similar to the self-similarity solution and distributed between the case of $M = 4$ and infinite shock strength. Then, to avoid confusion, the case of $M = 6$ is omitted. From these facts one finds that the difference between the finite and infinite shock strengths in the structure of a blast wave in a dust-laden gas is very small.

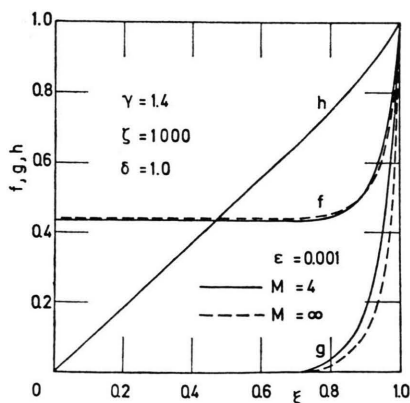


Fig. 4. The structure of blast waves for finite and infinite shock strength.

The effects of the volume fraction of the particles on the decay coefficients are shown in Figs. 5 and 6, respectively. Figure 5 shows the results for a given shock strength versus the volume fraction. The absolute value of the decay coefficients increases as the volume fraction increases. This means that the blast waves in a dusty gas decay rapidly and approach the flow of the infinite shock strength. The results for a fixed volume fraction of the particles are shown in Figure 6. The figure indicates that the decay coefficients decrease as the volume fraction increases and the absolute value of decay coefficients is nearly proportional to $1/M^2$.

The nondimensional density, velocity and pressure distributions behind the shock front for the case of $M = 4$ are shown in Figs. 7, 8 and 9, respectively. The concentration of density at the shock front becomes significant when the volume fraction

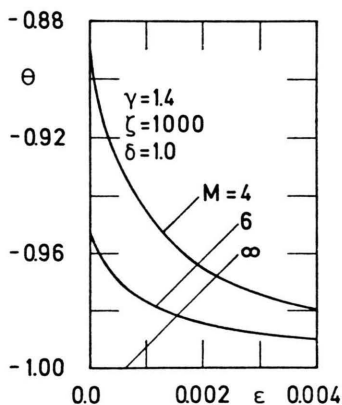


Fig. 5. The variation of decay coefficients with the volume fraction of particles.

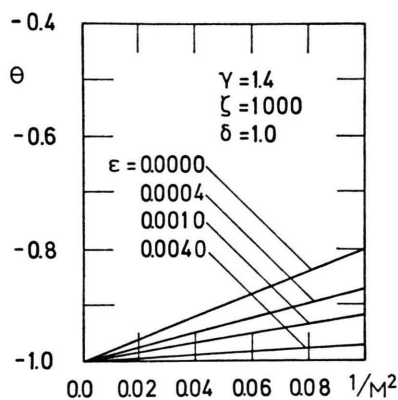


Fig. 6. The variation of decay coefficients with shock strength.

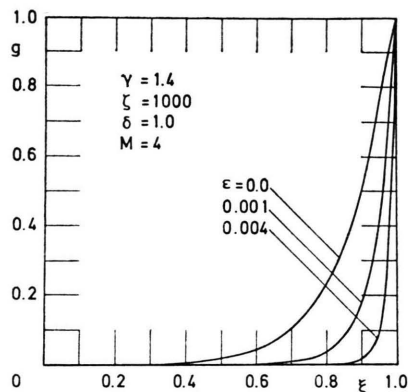


Fig. 7. The profiles of nondimensional density behind shock front for the first order solution.

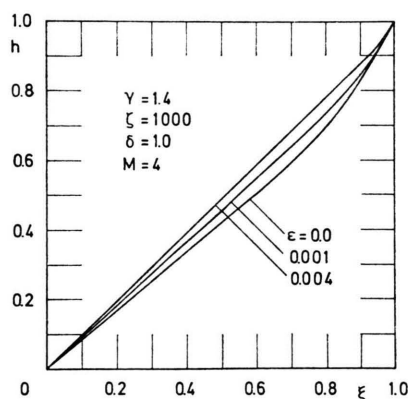


Fig. 8. The profiles of nondimensional velocity behind shock front for the first order solution.

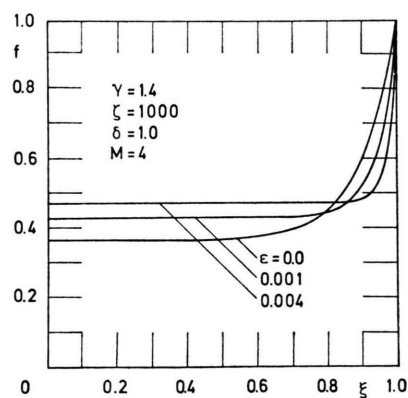


Fig. 9. The profiles of nondimensional pressure behind shock front for the first order solution.

of particles increases. This tendency is analogous to the case of self-similarity flow.

In the present work the analysis is restricted to the case of velocity and temperature equilibrium between the gas and particles. To make the analysis more realistic, one should take the nonequilibrium effect on the velocity and temperature into account. In addition, the flow temperature near the center of explosions is so high that the effect of phase changes between the flow media may play an important role. It is expected that the analysis on blast waves in a dusty gas will be extended to these complicated flows in the next step.

5. Conclusions

In the present paper the effect of suspended particles on blast waves is analysed by making use of a series expansion method with respect to $1/M^2$ under the assumption of velocity and temperature equilibrium. As a result one can conclude as follows:

As the volume fraction of the particles increases, the material density is assembled near the shock front and then the slopes of the pressure and velocity profiles increase. These tendencies can be seen quantitatively for both the zeroth and first order solutions. The difference between the zeroth and first order solutions becomes small when the volume fraction of the particles increases. The decay rate of a blast wave in a dusty gas is greater than that in a dust-free gas.

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- [1] M. A. Nettleton and R. Stirling, *Proc. Roy. Soc. A* **322**, 207 (1971).
- [2] M. Gilbert, J. Allport, and R. Dunlap, *ARS J.* **32**, 1929 (1962).
- [3] R. F. Probst and F. Fassio, *AIAA J.* **8**, 772 (1970).
- [4] T. Suzuki, F. Higashino, S. Oyagi, and A. Takano, The Initiation and Propagation of Spherical Detonation in a Particle-laden Gas. *Proc. 11th Int. Symp. Space Tech. and Scie.*, 65–70 (1975).
- [5] N. Gerber and J. M. Bartos, *AIAA J.* **12**, 120 (1974).
- [6] T. Suzuki, F. Higashino, and A. Takano, Blast Waves in a Dusty Gas, *Modern Developments in Shock Tube Research. Proc. 10th Int. Shock Tube Symp.*, 158–165 (1975).
- [7] G. Rudinger, *Phys. Fluids* **7**, 658 (1964).
- [8] A. Sakurai, *J. Phys. Soc. Japan* **8**, 662 (1953).